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# Graphical presentations of symmetric monoidal closed theories

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**Abstract.** We define a notion of symmetric monoidal closed (SMC) theory, consisting of a SMC signature augmented with equations, and describe the classifying categories of such theories in terms of proof nets.

## 1 Introduction

In this note, in preparation for a sequel using symmetric monoidal closed (SMC) categories to reconstruct Jensen and Milner’s [2004] bigraphs, we define a notion of SMC theory, and give a graphical presentation of the free SMC category generated by such a theory.

### 1.1 Symmetric monoidal closed theories

Recall that a *many-sorted algebraic theory* is specified by first giving a *signature*—a set of sorts  $X$  and a set  $\Sigma$  of operations with arities—together with a set of *equations* over that signature. For example, the theory for monoids is specified by taking only one sort  $x$ , and operations

$$m: x \times x \rightarrow x \quad \text{and} \quad e: 1 \rightarrow x,$$

together with the usual associativity and unitality equations. We may equally well view this signature as given by a graph

$$x \times x \xrightarrow{m} x \xleftarrow{e} 1$$

whose vertices are labelled by objects of the free category with finite products generated by  $X$ . In this paper, we follow the same route, but replacing from the start finite products with *symmetric monoidal closed* structure. Thus, an SMC signature is given by a set of sorts  $X$ , together with a graph with vertices in the free SMC category generated by  $X$ , so that instead of cartesian product, we have available the logical connectives of Girard [1987, 1993]’s Intuitionistic Multiplicative Linear Logic (henceforth IMLL): a tensor product  $\otimes$ , its right adjoint  $\multimap$ , and its unit  $I$ . This permits idioms from higher-order abstract syntax [Pfenning and Elliott, 1988], e.g., taking the graph

$$(x \multimap x) \xrightarrow{\lambda} x \xleftarrow{\circledast} (x \otimes x) \tag{1}$$

as a signature. An SMC theory is now given by a SMC signature, together with a set of equations over that signature. This notion of theory gives rise to a *functorial semantics* in the sense of Lawvere [1963], the crux of which is the following. We may define a notion of *model* for an SMC theory in an arbitrary SMC category, and may associate to each SMC theory  $\mathbb{T}$  a *classifying category*  $\mathcal{C}_{\mathbb{T}}$ : this being a small SMC category for which strict SMC functors  $\mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{D}$  are in bijection with models of  $\mathbb{T}$  in  $\mathcal{D}$ . The existence of  $\mathcal{C}_{\mathbb{T}}$  follows from general considerations of categorical universal algebra; but the description this gives of  $\mathcal{C}_{\mathbb{T}}$  is syntactic. The main purpose of this paper is to give a graphical presentation of  $\mathcal{C}_{\mathbb{T}}$ . Its objects will be IMLL formulae, while its morphisms are variants of Hughes' [2005] proof nets, satisfying a correctness criterion familiar from Danos and Regnier's [1989].

## 1.2 Related work

There is an extensive literature devoted to describing free SMC categories of the kind we consider here. In their seminal work on coherence for closed categories, Kelly and Mac Lane [1971] introduced what are now known as Kelly-MacLane graphs, but did not go so far as to obtain a characterisation of free SMC categories. Such a construction was first carried out by Trimble [1994], and subsequently Blute et al. [1996] and Tan [1997], using ideas taken from Girard's [1987, 1993] proof nets (actually, Blute et al. [1996] construct the free star-autonomous category, but the free SMC category is obtained as its full subcategory of IMLL formulae). Variations on this theme are presented by Lamarche and Strassburger [2006] and Hughes [2005]. In all cases, morphisms are roughly equivalence classes of proof nets, with variations in the presentation. In our sequel to this paper, we wish to make use of Hughes' presentation, mainly because:

- it reduces the graphical burden to the minimum: where others introduce nodes corresponding to linear logical connectives, Hughes does not;
- its composition behaves nicely: it is defined on representatives and given by a straightforward gluing of graphs, where others rely on tricky mechanisms, e.g., Trimble's [1994] *rewiring*.

On the other hand, Hughes' equivalence classes of proof nets have the inconvenience of lacking normal forms, which, e.g., Trimble's enjoy.

However, Hughes only construct the free SMC over a *set*, which merely accounts for the sorts of a signature. Thus we must extend his construction to deal with an arbitrary SMC theory, which we do by reducing from the general case to that of a free SMC on a set. Cheng [2003] observed a relationship between trees and Kelly-MacLane graphs, of which our result is essentially a generalisation.

## 2 Symmetric monoidal closed theories

Given a set  $X$ , we write  $\overline{X}$  for the set of symmetric monoidal closed (henceforth SMC) types over  $X$ ; it is inductively generated by the following grammar:

$$e ::= x \mid I \mid e \otimes e \mid e \multimap e \quad (\text{where } x \in X).$$

By a SMC *signature*, we mean a quadruple  $(X, \Sigma, s, t)$  where  $X$  is a set of ground types,  $\Sigma$  a set of ground terms, and  $s, t: \Sigma \rightarrow \overline{X}$  are source and target arity functions. We may also write  $\Sigma(a, b)$  for the set of  $f \in \Sigma$  for which  $s(f) = a$  and  $t(f) = b$ . For each SMC signature, we inductively generate the set  $\overline{\Sigma}$  of *derived terms*, together with source and target functions  $\overline{s}, \overline{t}: \overline{\Sigma} \rightarrow \overline{X}$ , as follows. We require that for each  $f \in \Sigma(a, b)$ , we have  $f \in \overline{\Sigma}(a, b)$ ; for each  $a, b, c \in \overline{X}$ , we have

$$\begin{aligned} \alpha_{abc} &\in \overline{\Sigma}(a \otimes (b \otimes c), (a \otimes b) \otimes c); & \alpha_{abc}^{-1} &\in \overline{\Sigma}((a \otimes b) \otimes c, a \otimes (b \otimes c)); \\ \lambda_a &\in \overline{\Sigma}(I \otimes a, a); & \lambda_a^{-1} &\in \overline{\Sigma}(a, I \otimes a); \\ \rho_a &\in \overline{\Sigma}(a \otimes I, a); & \rho_a^{-1} &\in \overline{\Sigma}(a, a \otimes I); \\ \sigma_{ab} &\in \overline{\Sigma}(a \otimes b, b \otimes a); & \epsilon_{ab} &\in \overline{\Sigma}((a \multimap b) \otimes a, b) \\ \text{and } \eta_{ab} &\in \overline{\Sigma}(a, b \multimap (a \otimes b)); \end{aligned}$$

for each  $f \in \overline{\Sigma}(a, b)$  and  $g \in \overline{\Sigma}(b, c)$ , we have  $g \circ f \in \overline{\Sigma}(a, c)$ ; for each  $a \in \overline{X}$ , we have  $\text{id}_a \in \overline{\Sigma}(a, a)$ ; and for each  $f \in \overline{\Sigma}(a, b)$  and  $g \in \overline{\Sigma}(c, d)$ , we have  $f \otimes g \in \overline{\Sigma}(a \otimes c, b \otimes d)$  and  $f \multimap g \in \overline{\Sigma}(c \multimap b, a \multimap d)$ . By an *equation* over a SMC signature, we mean a string of the form  $u = v: a \rightarrow b$  for some  $a, b \in \overline{X}$  and  $u, v \in \overline{\Sigma}(a, b)$ ; and by a *syntactic SMC theory* we mean an SMC signature  $(X, \Sigma)$  together with a set  $E$  of equations over it.

*Example 1.*

- The syntactic theory of monoids has a single ground sort  $x$ , ground terms  $e \in \Sigma(I, x)$  and  $m \in \Sigma(x \otimes x, x)$ , and three equations

$$\begin{aligned} (m \circ (m \otimes \text{id}_x)) \circ \alpha_{xxx} &= m \circ (\text{id}_x \otimes m): x \otimes (x \otimes x) \rightarrow x \\ m \circ (e \otimes \text{id}_x) &= \lambda_x: I \otimes x \rightarrow x \\ m \circ (\text{id}_x \otimes e) &= \rho_x: x \otimes I \rightarrow x. \end{aligned}$$

- The syntactic theory of the linear lambda-calculus has a single ground sort  $x$  and two terms,  $\lambda \in \Sigma(x \multimap x, x)$  and  $@ \in \Sigma(x \otimes x, x)$ . Its single equation is the  $\beta$ -rule

$$@ \circ (\lambda \otimes \text{id}_x) = \epsilon_{xx}: (x \multimap x) \otimes x \rightarrow x.$$

Given a syntactic theory  $\mathbb{T}$  and a SMC category  $\mathcal{D}$ , we may define a notion of *interpretation*  $F: \mathbb{T} \rightarrow \mathcal{D}$ . Such an  $F$  is given by a function  $F_X: X \rightarrow \text{ob } \mathcal{D}$  interpreting the ground types of the theory, together with a family of functions

$$F_{a,b}: \Sigma(a, b) \rightarrow \mathcal{D}(\overline{F_X}(a), \overline{F_X}(b)) \quad (\text{for } a, b \in \overline{X})$$

interpreting the basic terms; here we write  $\overline{F_X}$  for the unique extension of  $F_X$  to a function  $\overline{X} \rightarrow \text{ob } \mathcal{D}$  commuting with the SMC type constructors. These data are required to satisfy each of the equations of the theory, in the sense that

$$u = v: a \rightarrow b \text{ in } E \quad \Rightarrow \quad \overline{F_{a,b}}(u) = \overline{F_{a,b}}(v): \overline{F_X}(a) \rightarrow \overline{F_X}(b) \text{ in } \mathcal{D}.$$

Here  $\overline{F_{a,b}}$  denotes the unique extension of  $F_{a,b}$  to a function  $\overline{\Sigma}(a, b) \rightarrow \mathcal{D}(\overline{F_X}(a), \overline{F_X}(b))$  commuting with the SMC term constructors.

*Example 2.*

- An interpretation in  $\mathcal{D}$  of the theory of monoids is a monoid in  $\mathcal{D}$ .
- An interpretation in  $\mathcal{D}$  of the theory of the linear lambda-calculus is given by an object  $X \in \mathcal{D}$  and maps  $\lambda: X \multimap X \rightarrow X$  and  $@: X \otimes X \rightarrow X$  rendering commutative the diagram

$$\begin{array}{ccc} (X \multimap X) \otimes X & \xrightarrow{\lambda \otimes X} & X \otimes X \\ & \searrow \epsilon_{X, X} & \downarrow @ \\ & & X. \end{array}$$

*Property 1.* To each syntactic theory  $\mathbb{T} = (X, \Sigma, E)$  we may assign a small SMC category  $\mathcal{C}_{\mathbb{T}}$  which classifies  $\mathbb{T}$ , in the sense that there is a bijection, natural in  $\mathcal{D}$ , between interpretations  $\mathbb{T} \rightarrow \mathcal{D}$  and strict SMC functors  $\mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{D}$ .

*Proof.* We take the set of objects of  $\mathcal{C}_{\mathbb{T}}$  to be  $\overline{X}$ , and obtain its homsets by quotienting the sets  $\overline{\Sigma}(a, b)$  under the smallest congruence which contains each equation in  $E$ ; makes composition associative and unital; makes  $\otimes$  and  $\multimap$  functorial in each variable; makes  $\alpha, \lambda, \rho, \sigma, \epsilon$  and  $\eta$  natural in each variable; makes the  $\lambda^{-1}$ 's,  $\rho^{-1}$ 's and  $\alpha^{-1}$ 's inverse to the  $\lambda$ 's,  $\rho$ 's and  $\alpha$ 's; verifies the triangle identities for  $\eta$  and  $\epsilon$ ; and verifies the symmetric monoidal category axioms of Mac Lane.

Observe that different syntactic theories  $\mathbb{T}$  and  $\mathbb{T}'$  may give rise to the same classifying category  $\mathcal{C}_{\mathbb{T}} = \mathcal{C}_{\mathbb{T}'}$ , and so have the same models. Thus, in the spirit of categorical logic, one should view syntactic SMC theories as *presentations* of their classifying categories; so that to understand a syntactic theory  $\mathbb{T}$  is really to understand the category  $\mathcal{C}_{\mathbb{T}}$ . The purpose of this note is to improve this understanding by giving a *graphical representation* of  $\mathcal{C}_{\mathbb{T}}$ , in which morphisms are viewed as certain equivalence classes of diagrams. In the case where our theory has no equations, and our signature no operations, we are considering a mere set of types  $X$ , and the corresponding SMC category  $\mathcal{C}_X$  is the free SMC category on  $X$ . We have mentioned that in this case we want to use Hughes' [2005] representation. We will show that this special case suffices to derive the general one. In fact, it will suffice to derive the case of a *free theory*—one given by a signature  $(X, \Sigma)$  subject to no equations—since the classifying category of an arbitrary theory may be obtained by quotienting out the morphisms of the classifying category of a free theory, so that a graphical representation of the latter induces a graphical representation of the former.

Given a free theory  $(X, \Sigma)$ , we will obtain a graphical representation of the corresponding classifying category  $\mathcal{C}_{X, \Sigma}$  by first describing it in terms of  $\mathcal{C}_X$ , the free SMC category on  $X$ , and then making use of a suitable graphical description of the latter. We begin by introducing some notation. We define the *typing function*  $\text{ty}: \Sigma \rightarrow \text{ob } \mathcal{C}_X = \overline{X}$  by  $\text{ty}(\alpha) = s(\alpha) \multimap t(\alpha)$ , and extend this to a

function on  $\Sigma^*$ , the set of lists in  $\Sigma$ , by taking

$$\begin{aligned} \mathbf{ty}() &= I, & \mathbf{ty}(\alpha) &= s(\alpha) \multimap t(\alpha), \\ \text{and } \mathbf{ty}(\alpha_1, \dots, \alpha_n) &= \mathbf{ty}(\alpha_1, \dots, \alpha_{n-1}) \otimes \mathbf{ty}(\alpha_n) \text{ for } n \geq 2. \end{aligned}$$

Though we may not have equality between  $\mathbf{ty}(\alpha_1, \dots, \alpha_n) \otimes \mathbf{ty}(\beta_1, \dots, \beta_m)$  and  $\mathbf{ty}(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m)$ , we can at least build a canonical isomorphism between them in  $\mathcal{C}_X$  using the associativity and unitality constraints. Similarly, for  $\sigma$  a permutation on  $n$  letters, we can construct canonical maps

$$\hat{\sigma}: \mathbf{ty}(\alpha_1, \dots, \alpha_n) \rightarrow \mathbf{ty}(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)})$$

using the symmetry isomorphisms of  $\mathcal{C}_X$ . We now define a category  $\mathcal{C}'_{X,\Sigma}$  of which the classifying category  $\mathcal{C}_{X,\Sigma}$  will be a quotient.

- **Objects** are objects of  $\mathcal{C}_X$ ;
- **Morphisms**  $U \rightarrow V$  are given by a list  $\Gamma \in \Sigma^*$  together with a morphism

$$\phi: \mathbf{ty}(\Gamma) \otimes U \rightarrow V$$

in  $\mathcal{C}_X$ .

- **Identity maps**  $U \rightarrow U$  are given by the empty list  $()$  together with the canonical isomorphism  $I \otimes U \rightarrow U$ ;
- **Composition** of maps  $(\Gamma, \phi): U \rightarrow V$  and  $(\Delta, \psi): V \rightarrow W$  is given by the map  $(\Delta + \Gamma, \xi): U \rightarrow W$ , wherein  $\Delta + \Gamma$  is the concatenation of the two lists, and  $\xi$  is the composite morphism

$$\begin{array}{ccc} \mathbf{ty}(\Delta + \Gamma) \otimes U & & W \\ \cong \downarrow & & \uparrow \psi \\ \mathbf{ty}(\Delta) \otimes (\mathbf{ty}(\Gamma) \otimes U) & \xrightarrow{\mathbf{ty}(\Delta) \otimes \phi} & \mathbf{ty}(\Delta) \otimes V \end{array}$$

The category  $\mathcal{C}'_{X,\Sigma}$  admits an embedding functor  $i: \mathcal{C}_X \rightarrow \mathcal{C}'_{X,\Sigma}$ , which is the identity on objects, and on morphisms sends a map  $\phi: U \rightarrow V$  to the pair of the empty list  $()$  together with the composite

$$I \otimes U \xrightarrow{\cong} U \xrightarrow{\phi} V.$$

It also admits a tensor operation, which on objects is inherited from  $\mathcal{C}_X$ ; and on morphisms takes a pair of maps  $(\Gamma, \phi): U \rightarrow V$  and  $(\Gamma', \phi'): U' \rightarrow V'$  to the map  $(\Gamma + \Gamma', \theta): U \otimes U' \rightarrow V \otimes V'$ , where  $\theta$  is the composite

$$\begin{array}{ccc} \mathbf{ty}(\Gamma + \Gamma') \otimes (U \otimes U') & & V \otimes V' \\ \cong \searrow & \xrightarrow{\phi \otimes \phi'} & \\ & (\mathbf{ty}(\Gamma) \otimes U) \otimes (\mathbf{ty}(\Gamma') \otimes U') & \end{array}$$

However, this tensor operation does not underlie a tensor product in the usual sense; for whilst functorial in each variable separately, it does not satisfy the

compatibility conditions required to obtain a functor of two variables. These require the commutativity of squares of the form

$$\begin{array}{ccc} U \otimes U' & \xrightarrow{f \otimes U'} & V \otimes U' \\ U \otimes f' \downarrow & & \downarrow V \otimes f' \\ U \otimes V' & \xrightarrow{f \otimes V'} & V \otimes V'; \end{array} \quad (2)$$

but we see from the definitions that, for  $f = (\Gamma, \phi)$  and  $f' = (\Gamma', \phi')$  as above, the upper composite in (2) has its first component given by  $\Gamma' + \Gamma$ , whilst the lower has it given by  $\Gamma + \Gamma'$ ; so that  $\mathcal{C}'_{X,\Sigma}$  is not a SMC category. Nonetheless, we do have that:

*Property 2.*  $\mathcal{C}'_{X,\Sigma}$  is a symmetric premonoidal category in the sense of Power and Robinson [1997], and the embedding  $i: \mathcal{C}_X \rightarrow \mathcal{C}'_{X,\Sigma}$  is a strict symmetric premonoidal functor.

*Proof.* Beyond the structure we have already noted, this means that  $\mathcal{C}'_{X,\Sigma}$  comes equipped with a unit object, which we take to be  $I$ , the unit object of  $\mathcal{C}_X$ ; and with isomorphisms of associativity, unitality and symmetry of the same form as those for a symmetric monoidal category, but differing from them in two aspects. First, they need only be natural in each variable separately; so for symmetry, for instance, we only require diagrams of the following form to commute:

$$\begin{array}{ccc} U \otimes V & \xrightarrow{U \otimes g} & U \otimes V' \\ \sigma_{U,V} \downarrow & & \downarrow \sigma_{U,V'} \\ V \otimes U & \xrightarrow{g \otimes U} & V' \otimes U \end{array} \quad \text{and} \quad \begin{array}{ccc} U \otimes V & \xrightarrow{f \otimes V} & U' \otimes V \\ \sigma_{U,V} \downarrow & & \downarrow \sigma_{U',V} \\ V \otimes U & \xrightarrow{V \otimes f} & V \otimes U'. \end{array}$$

Secondly, the constraint isomorphisms are required to be *central* maps, where  $f: U \rightarrow V$  is said to be *central* just when for each  $f': U' \rightarrow V'$ , the diagram (2) and its dual

$$\begin{array}{ccc} U' \otimes U & \xrightarrow{U' \otimes f} & U' \otimes V \\ f' \otimes U \downarrow & & \downarrow f' \otimes V \\ V' \otimes U & \xrightarrow{V' \otimes f} & V' \otimes V \end{array}$$

are rendered commutative. In the case of  $\mathcal{C}'_{X,\Sigma}$ , we fulfil these demands by taking each coherence constraint in  $\mathcal{C}'_{X,\Sigma}$  to be the image of the corresponding coherence constraint in  $\mathcal{C}_X$  under  $i: \mathcal{C}_X \rightarrow \mathcal{C}'_{X,\Sigma}$ . Naturality in each variable is easily checked; whilst centrality follows by observing that a map of  $\mathcal{C}'_{X,\Sigma}$  is central iff it lies in the image of the aforementioned embedding. Finally, we observe that the embedding  $i: \mathcal{C}_X \rightarrow \mathcal{C}'_{X,\Sigma}$  preserves all the structure of  $\mathcal{C}_X$  on the nose, and sends central maps to central maps; and so is strict symmetric premonoidal.

In fact,  $\mathcal{C}'_{X,\Sigma}$  is *closed* as a premonoidal category in the sense that for each  $V \in \mathcal{C}'_{X,\Sigma}$ , the endofunctor  $(-) \otimes V$  has a right adjoint  $V \multimap (-)$  which preserves central maps, with the units and counits

$$U \longrightarrow V \multimap (U \otimes V) \quad \text{and} \quad (V \multimap W) \otimes V \longrightarrow W$$

of these adjunctions being central. Indeed, we may take the action of  $V \multimap (-)$  on objects to be given as in  $\mathcal{C}_X$ ; and then we have:

$$\begin{aligned} \mathcal{C}'_{X,\Sigma}(U \otimes V, W) &= \coprod_{\Gamma \in \Sigma^*} \mathcal{C}_X(\text{ty}(\Gamma) \otimes (U \otimes V), W) \\ &\cong \coprod_{\Gamma \in \Sigma^*} \mathcal{C}_X(\text{ty}(\Gamma) \otimes U, V \multimap W) \\ &= \mathcal{C}'_{X,\Sigma}(U, V \multimap W), \end{aligned}$$

naturally in  $U$  and  $W$ , as desired. The centrality requirements now amount to the fact that the adjunctions

$$(-) \otimes V \dashv V \multimap (-): \mathcal{C}'_{X,\Sigma} \rightarrow \mathcal{C}'_{X,\Sigma}$$

may be restricted and corestricted to adjunctions

$$(-) \otimes V \dashv V \multimap (-): \mathcal{C}_X \rightarrow \mathcal{C}_X.$$

The reason that  $\mathcal{C}'_{X,\Sigma}$  is only premonoidal rather than monoidal is that its morphisms are built from a *list*, rather than a *multiset* of generating operations: in computational terms, we may think that a morphism “remembers the order in which its generating operations are executed”. To rectify this, we quotient out the morphisms of  $\mathcal{C}'_{X,\Sigma}$  by the action of the symmetric groups; the result will be the SMC category  $\mathcal{C}_{X,\Sigma}$  we seek. So let there be given a list  $\Gamma = (\alpha_1, \dots, \alpha_n) \in \Sigma^*$ , a permutation  $\sigma \in S_n$ , and a morphism  $\phi: \text{ty}(\sigma\Gamma) \otimes U \rightarrow V$  in  $\mathcal{C}_X$ , where  $\sigma\Gamma$  is the list  $(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)})$ . A generating element for our congruence  $\sim$  on the morphisms of  $\mathcal{C}'_{X,\Sigma}$  is now given by

$$(\sigma\Gamma, \phi) \sim (\Gamma, \phi \circ (\hat{\sigma} \otimes U))$$

where we recall that  $\hat{\sigma}$  is the canonical morphism  $\text{ty}(\Gamma) \rightarrow \text{ty}(\sigma\Gamma)$  built from symmetry and associativity maps in  $\mathcal{C}_X$ . We may now verify that for morphisms

$$U \xrightarrow{f} V \xrightleftharpoons[h]{g} W \xrightarrow{k} Z$$

in  $\mathcal{C}'_{X,\Sigma}$ ,  $g \sim h$  implies both  $gf \sim hf$  and  $kg \sim kh$ , so that  $\sim$  is a congruence on  $\mathcal{C}'_{X,\Sigma}$ , and we may define the category  $\mathcal{C}_{X,\Sigma}$  to be the quotient of  $\mathcal{C}'_{X,\Sigma}$  by  $\sim$ .

*Property 3.*  $\mathcal{C}_{X,\Sigma}$  is a symmetric monoidal closed category, and the quotient map  $q: \mathcal{C}'_{X,\Sigma} \rightarrow \mathcal{C}_{X,\Sigma}$  is a strict symmetric premonoidal functor.



*Proof.* Straightforward checking shows that if  $f \sim f'$  and  $g \sim g'$  in  $\mathcal{C}'_{X,\Sigma}$ , then  $f \otimes g \sim f' \otimes g'$ , so that the tensor operation on  $\mathcal{C}'_{X,\Sigma}$  passes to the quotient  $\mathcal{C}_{X,\Sigma}$ . For this operation to define a bifunctor on  $\mathcal{C}_{X,\Sigma}$ , we must verify that squares of the form (2) commute in  $\mathcal{C}_{X,\Sigma}$ : and this follows by checking that

$$(f \otimes V') \circ (U \otimes f') \sim (V \otimes f') \circ (f \otimes U')$$

in  $\mathcal{C}'_{X,\Sigma}$ . This defines our binary tensor on  $\mathcal{C}_{X,\Sigma}$ ; whilst the nullary tensor we inherit from  $\mathcal{C}'_{X,\Sigma}$ . The associativity, unitality and symmetry constraints in the category  $\mathcal{C}_{X,\Sigma}$  are obtained as the image of the corresponding constraints in  $\mathcal{C}'_{X,\Sigma}$  under the quotient map. Commutativity of the triangle, pentagon and hexagon axioms is inherited; whilst the (restricted) naturality of these maps in  $\mathcal{C}'_{X,\Sigma}$  becomes their (full) naturality in  $\mathcal{C}_{X,\Sigma}$ . Thus  $\mathcal{C}_{X,\Sigma}$  is symmetric monoidal. It is now easy to check that the isomorphisms  $\mathcal{C}'_{X,\Sigma}(U \otimes V, W) \cong \mathcal{C}'_{X,\Sigma}(U, V \multimap W)$  descend along the quotient map, and so induce a closed structure on  $\mathcal{C}_{X,\Sigma}$ . Finally, since each piece of structure on  $\mathcal{C}_{X,\Sigma}$  is obtained from the corresponding piece of structure on  $\mathcal{C}'_{X,\Sigma}$ , the quotient map  $q: \mathcal{C}'_{X,\Sigma} \rightarrow \mathcal{C}_{X,\Sigma}$  is strict symmetric premonoidal as required.

Observe that the composite functor  $qi: \mathcal{C}_X \rightarrow \mathcal{C}_{X,\Sigma}$ , is a strict symmetric premonoidal closed functor between two symmetric monoidal closed categories; and as such, is actually a strict symmetric *monoidal* closed functor. We make use of this fact below.

**Theorem 1.**  $\mathcal{C}_{X,\Sigma}$  is the classifying category of the syntactic theory with signature  $(X, \Sigma)$  and no equations.

*Proof.* Suppose first given a strict SMC functor  $F: \mathcal{C}_{X,\Sigma} \rightarrow \mathcal{D}$ ; we obtain an interpretation  $G: (X, \Sigma) \rightarrow \mathcal{D}$  by taking

$$G_X(x) = F(x) \quad \text{and} \quad G_{a,b}(\alpha) = F[\alpha]: Fa \rightarrow Fb, \quad (3)$$

where, for  $\alpha \in \Sigma(a, b)$ , the morphism  $[\alpha]: a \rightarrow b$  of  $\mathcal{C}_{X,\Sigma}$  is given by  $q((\alpha), \epsilon_{ab})$ . Conversely, we must show that each interpretation  $G: (X, \Sigma) \rightarrow \mathcal{D}$  lifts to a unique strict SMC functor  $F: \mathcal{C}_{X,\Sigma} \rightarrow \mathcal{D}$  satisfying (3). The action of  $G$  on ground types is given by a function  $G_X: X \rightarrow \text{ob } \mathcal{D}$ ; and this is equally well a functor  $G_X: X \rightarrow \mathcal{D}$ —with  $X$  regarded now as a discrete category—which, as  $\mathcal{C}_X$  is the free SMC category on  $X$ , lifts to a strict SMC functor  $\tilde{G}_X: \mathcal{C}_X \rightarrow \mathcal{D}$ . It follows that  $F$ , if it exists, must makes the following diagram of strict SMC functors commute:

$$\begin{array}{ccc} & \mathcal{C}_{X,\Sigma} & \\ qi \nearrow & & \searrow F \\ \mathcal{C}_X & \xrightarrow{\tilde{G}_X} & \mathcal{D} \end{array} \quad (4)$$

Indeed, to ask that the first equation in (3) should hold is equally well to ask that (4) should commute when precomposed with the functor  $\eta: X \rightarrow \mathcal{C}_X$  exhibiting  $\mathcal{C}_X$  as free on  $X$ ; and by the uniqueness part of the universal property of

$\mathcal{C}_X$ , this is equally well to ask (4) itself to commute. This determines the action of  $F$  on the objects and certain of the morphisms of  $\mathcal{C}_{X,\Sigma}$ ; let us now extend this to deal with an arbitrary morphism  $f: U \rightarrow V$ . If  $f$  is represented by some  $(\Gamma, \phi)$  in  $\mathcal{C}'_{X,\Sigma}$ , then we may factorise it as

$$U \xrightarrow{q(\Gamma, \text{id})} \text{ty}(\Gamma) \otimes U \xrightarrow{q(i(\phi))} V$$

in  $\mathcal{C}_{X,\Sigma}$ ; and commutativity in (4) forces  $F$ , if it exists, to send the second part of this factorisation to  $\tilde{G}_X(\phi)$ . For the first part, either we have  $\Gamma$  empty, in which case  $q(\Gamma, \text{id})$  is the unit isomorphism  $U \cong I \otimes U$ ; or we have  $\Gamma = (\alpha_1, \dots, \alpha_n)$ , in which case  $q(\Gamma, \text{id})$  decomposes as

$$U \xrightarrow{\cong} \left( \bigotimes_{1 \leq i \leq n} I \right) \otimes U \xrightarrow{\bigotimes_i \overline{[\alpha_i]} \otimes U} \text{ty}(\Gamma) \otimes U,$$

where  $\overline{[\alpha_i]}: I \rightarrow \text{ty}(\alpha_i)$  is the exponential transpose of  $[\alpha_i]: s(\alpha_i) \rightarrow t(\alpha_i)$  in  $\mathcal{C}_{X,\Sigma}$ . But since we require  $F$ , if it exists, to both satisfy the second equation in (3) and strictly preserve the SMC structure, this determines its value on  $q(\Gamma, \text{id})$ ; and hence on an arbitrary morphism of  $\mathcal{C}_{X,\Sigma}$ . Consequently, there is at most one strict SMC functor  $F: \mathcal{C}_{X,\Sigma} \rightarrow \mathcal{C}$  satisfying the equations in (3); and in order to conclude that there is *exactly* one such, we must check that the assignments described above underlie a well-defined strict SMC functor  $F$ . This follows by straightforward calculation: as a representative sample of which, we verify that  $F$  as given above is well-defined on morphisms. So let there be given  $f: U \rightarrow V$  in  $\mathcal{C}_{X,\Sigma}$ , together with two morphisms  $(\sigma\Gamma, \phi)$  and  $(\Gamma, \phi \circ (\hat{\sigma} \otimes U))$  of  $\mathcal{C}'_{X,\Sigma}$  which represent it. Then we have the following commutative diagram in  $\mathcal{C}_{X,\Sigma}$ :

$$\begin{array}{ccccc} & & \text{ty}(\Gamma) \otimes U & & \\ & \nearrow q(\Gamma, \text{id}) & \downarrow q(i(\hat{\sigma} \otimes U)) & \nwarrow q(i(\phi \circ (\hat{\sigma} \otimes U))) & \\ U & & & & V \\ & \searrow q(\sigma\Gamma, \text{id}) & \downarrow q(i(\phi)) & \nearrow & \\ & & \text{ty}(\sigma\Gamma) \otimes U & & \end{array}$$

and must show that the corresponding diagram commutes when we apply  $F$ . This is clear for the right-hand triangle; whilst for the left-hand one, it amounts to checking the following equality in  $\mathcal{D}$ :

$$\begin{array}{ccc} & FU & \\ \cong \swarrow & & \searrow \cong \\ \left( \bigotimes_{1 \leq i \leq n} I \right) \otimes FU & & \left( \bigotimes_{1 \leq i \leq n} I \right) \otimes FU \\ \downarrow \bigotimes_i \overline{[\alpha_i]} \otimes FU & & \downarrow \bigotimes_i \overline{[\alpha_{\sigma i}]} \otimes FU \\ F(\text{ty}(\Gamma)) \otimes FU & \xrightarrow{F\hat{\sigma} \otimes FU} & F(\text{ty}(\sigma\Gamma)) \otimes FU, \end{array}$$

which follows immediately from the symmetric monoidal closed category axioms. The remaining calculations proceed similarly.

Finally in this section, we consider the case of a general theory  $\mathbb{T} = (X, \Sigma, E)$ . Let  $\mathcal{C}_{\mathbb{T}}$  be the quotient of  $\mathcal{C}_{X, \Sigma}$  by the smallest congruence  $\sim$  which contains all the equations in  $E$  and respects the SMC structure. We have:

**Theorem 2.**  $\mathcal{C}_{\mathbb{T}}$  is the classifying category of the theory  $\mathbb{T}$ .

In fact, using a linear analogue of Lambek and Scott's [1988] *functional completeness*, we may give a more direct characterisation of the congruence  $\sim$ . Here we write  $\ulcorner f \urcorner: I \rightarrow a \multimap b$  to denote the currying of any map  $f: a \rightarrow b$ .

*Property 4.* We obtain  $\sim$  as the smallest equivalence relation generated by  $\sim_1$ , where  $f \sim_1 g: a \rightarrow b$  just when there exists an equation  $u = v: c \rightarrow d$  in  $E$  and map  $h$  such that  $f$  is

$$a \xrightarrow{\cong} I \otimes a \xrightarrow{\ulcorner u \urcorner \otimes a} (c \multimap d) \otimes a \xrightarrow{h} b$$

and replacing  $u$  with  $v$  yields  $g$ .

### 3 A graphical representation of the classifying category

Putting Theorem 1 together with Hughes [2005]'s graphical description of  $\mathcal{C}_X$ , we obtain the following graphical representation of the category  $\mathcal{C}_{X, \Sigma}$ . First, for each type  $a \in \overline{X}$ , we define the *ports* of  $a$  to be the set of leaf occurrences in it, which may either be of type  $I$ , or of ground types  $x \in X$ . Ports are signed *positive* when they are reached by passing to the left of an even number of  $\multimap$ , and *negative* otherwise. We let  $a^+$  and  $a^-$  denote the sets of positive and negative ports of  $a$ , respectively. We define a *support* to be a finite set labelled by elements of  $\Sigma$ . The *ports* of a support  $C$  are defined by

$$C^+ = \coprod_{c \in C} (\text{ty}(\alpha_c))^+ \quad \text{and} \quad C^- = \coprod_{c \in C} (\text{ty}(\alpha_c))^- ,$$

where  $\alpha_c$  is the label of  $c$ . We now define the category  $\mathcal{D}_{X, \Sigma}^0$  of  $(X, \Sigma)$ -prenets to have:

- **Objects** being elements of  $\overline{X}$ .
- **Morphisms**  $a \rightarrow b$  being given by a support  $C$  together with a directed graph  $G$ , whose vertices are the disjoint union of the ports of  $a$ ,  $b$  and  $C$ ; and whose edges are such that the incidence relation is the graph of a partial function

$$g: a^+ + C^+ + b^- \rightarrow a^- + C^- + b^+, \quad (5)$$

that restricts to a bijection of  $x$ -labeled ports for each  $x \in X$ . We consider morphisms equivalent up to the choice of support (replacing  $C$  with isomorphic  $C'$ , preserving  $g$ ).

- **Identity maps**  $a \rightarrow a$  being given by the empty support together with the identity graph.

- **Composition** of maps  $(C, G): a \rightarrow b$  and  $(D, H): b \rightarrow c$  being given by the map  $(C + D, G +_b H): a \rightarrow c$ , where  $G +_b H$  is obtained by glueing the graphs  $G$  and  $H$  together along the ports of  $b$ . More formally, if  $x \in G$  and  $z \in H$ , then  $G +_b H$  will have an edge  $x \rightarrow z$  whenever there exist ports  $y_1, \dots, y_k$  of  $b$  and edges

$$\begin{array}{ccccccc} x \longrightarrow y_1 & y_2 \longrightarrow y_3 & \cdots & y_{k-1} \longrightarrow y_k & & & \text{in } G \\ \text{and} & y_1 \longrightarrow y_2 & y_3 \longrightarrow & \cdots & \longrightarrow y_{k-1} & y_k \longrightarrow z & \text{in } H. \end{array}$$

There are three analogous cases when:

- $x \in H$  and  $z \in G$ ,
- $x, z \in H$ , or
- $x, z \in G$ .

We now consider the subcategory  $\mathcal{D}_{X, \Sigma}^1$  of  $(X, \Sigma)$ -nets with the same objects, but whose morphisms are *correct* prenets in the following sense. First, for any IMLL formula  $a$ , let  $a'$  be its representation in classical MLL, i.e., using  $\otimes$ ,  $\wp$ ,  $I$ ,  $\perp$ , and signed ground types  $x$  and  $x^\perp$ ; in particular,  $(a \multimap b)' = a'^\perp \wp b'$ . Now by a *switching* of  $a'$ , we mean a graph obtained by cutting exactly one premise of each  $\wp$  node in the abstract syntax tree of  $a'$ ; and by a *switching* of a  $(X, \Sigma)$ -prenet  $(C, G): a \rightarrow b$ , we mean a graph obtained by glueing along the ports:

- A switching of  $a'^\perp$ ;
- A switching of  $b'$ ;
- A switching of each  $\alpha_c'^\perp$  (where  $\alpha_c$  is the label of  $c \in C$ ); and
- The graph  $G$  (forgetting the orientation).

The prenet  $(C, G)$  is said to be *correct*, or a *net*, just when all its switchings are trees. The nets  $a \rightarrow b$  are in close correspondence with the morphisms  $a \rightarrow b$  in the free SMC category  $\mathcal{C}_{X, \Sigma}$ . To see this, suppose given a net  $(C, G): a \rightarrow b$  whose support is a finite set  $\{1, \dots, n\}$ . If we define  $\Gamma = (\alpha_1, \dots, \alpha_n)$  then we have  $C^+ = \coprod_{1 \leq i \leq n} (\text{ty}(\alpha_i))^+ \cong (\text{ty}(\Gamma))^+$  and  $C^- \cong (\text{ty}(\Gamma))^-$ ; and we claim that the composite partial function

$$\begin{array}{ccc} (\text{ty}(\Gamma) \otimes a)^+ + b^- & \xrightarrow{\cong} & a^+ + C^+ + b^- \\ \downarrow g' & & \downarrow g \\ (\text{ty}(\Gamma) \otimes a)^- + b^+ & \xleftarrow{\cong} & a^- + C^- + b^+. \end{array}$$

describes a morphism  $\text{ty}(\Gamma) \otimes a \rightarrow b$  in Hughes [2005]’s presentation of the free SMC category  $\mathcal{C}_X$  over  $X$ . For this, we just have to show correctness; but any switching of  $(\text{ty}(\Gamma) \otimes a)^\perp$  amounts to a disjoint union of a switching of each of  $a^\perp$  and the  $\text{ty}(\alpha_i)^\perp$ ’s, so that correctness follows from that of  $g$ . Thus  $(C, G)$  yields a morphism  $a \rightarrow b$  in  $\mathcal{C}_{X, \Sigma}$ ; and conversely, given  $\Gamma$ , any correct representative

$g'$  in the sense of Hughes defines a correct net in our sense, with reordering of  $\Gamma$  resulting in an isomorphism of supports.

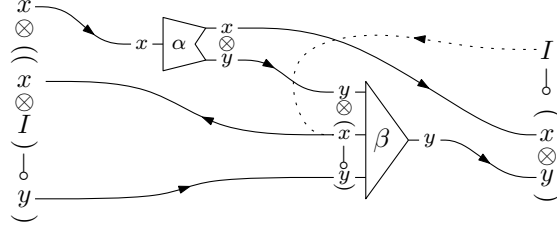
Finally, we may mimic Trimble rewiring in our setting: say that  $f \sim g$  when  $g$  is obtained by changing the target of a single edge from a negative occurrence of  $I$  in  $f$ , preserving correctness. This extends to an equivalence relation which we call *rewiring*. Letting  $\mathcal{D}_{X,\Sigma}$  be the quotient of  $\mathcal{D}_{X,\Sigma}^1$  modulo rewiring, we obtain:

**Theorem 3.** *The categories  $\mathcal{D}_{X,\Sigma}$  and  $\mathcal{C}_{X,\Sigma}$  are isomorphic in SMCCat.*

The category  $\mathcal{D}_{X,\Sigma}$  provides a graphical representation of the free SMC category generated by  $(X, \Sigma)$ . If  $X = \{x, y\}$  and  $\Sigma$  is described by the following graph:

$$x \xrightarrow{\alpha} x \otimes y \quad y \otimes (x \multimap y) \xrightarrow{\beta} y$$

then an example morphism from  $x \otimes ((x \otimes I) \multimap y)$  to  $I \multimap (x \otimes y)$  of  $\mathcal{D}_{X,\Sigma}$  is:



Notice that the dotted link can be rewired to any positive port.

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